Problem 1

See the solutions to PS#1 from last year.

Problem 2

Part (a)

- We first prove that L^p is a vector space.

  Let $f, g \in L^p$. Since $f : [0,1] \to \mathbb{R}$, $g : [0,1] \to \mathbb{R}$, $f + g : [0,1] \to \mathbb{R}$.

  Note that $|f + g|^p \le |2 \max\{f, g\}|^p = 2^p |\max\{f, g\}|^p \le 2^p(|f|^p + |g|^p)$.

  Then $\int_0^1 |f(x) + g(x)|^p dx \le 2^p \int_0^1 |f(x) + g(x)|^p dx + 2^p \int_0^1 |g(x)|^p dx < \infty$.

  Thus $f + g \in L^p$.

  Now, let $f \in L^p$ and $\alpha \in \mathbb{R}$. Since $f : [0,1] \to \mathbb{R}$, $\alpha f : [0,1] \to \mathbb{R}$. $\int_0^1 |\alpha f(x)|^p dx = \int_0^1 |\alpha|^p \int_0^1 |f(x)|^p dx \le \infty$. Thus $\alpha f \in L^p$.

  Therefore, L^p is a vector space.

- We now prove that $(\int_0^1 |f(x)|^p dx)^{(1/p)} \forall f \in L^p$ is a norm.

  1. Show (a) $||f||_p \ge 0$ (b) $||f||_p = 0 \iff f = 0$

     (a) $\forall x \in [0,1]$ $|f(x)| \ge 0 \Rightarrow |f(x)|^p \ge 0$ $\forall x \in [0,1] \Rightarrow \int_0^1 |f(x)|^p dx \ge 0$
     $\Rightarrow (\int_0^1 |f(x)|^p dx)^{(1/p)} \ge 0 \Rightarrow ||f||_p \ge 0$

     (b) $||f||_p = 0 \Rightarrow (\int_0^1 |f(x)|^p dx)^{(1/p)} = 0 \Rightarrow \int_0^1 |f(x)|^p dx = 0 \Rightarrow |f(x)| = 0$ $\forall x \in [0,1]$
     $\Rightarrow f(x) = 0$ $\forall x \in [0,1] \Rightarrow f = 0$
     $f = 0 \Rightarrow f(x) = 0$ $\forall x \in [0,1] \Rightarrow |f(x)| = 0$ $\forall x \in [0,1] \Rightarrow |f(x)|^p = 0$ $\forall x \in [0,1]$
     $\Rightarrow \int_0^1 |f(x)|^p dx = 0 \Rightarrow (\int_0^1 |f(x)|^p dx)^{(1/p)} = 0 \Rightarrow ||f||_p = 0.$

  2. $||\alpha f||_p = (\int_0^1 |\alpha f(x)|^p dx)^{(1/p)} = (\int_0^1 |\alpha|^p |f(x)|^p dx)^{(1/p)}$
     $= (|\alpha|^p \int_0^1 |f(x)|^p dx)^{(1/p)} = |\alpha| (\int_0^1 |f(x)|^p dx)^{(1/p)} = |\alpha| ||f||_p$

  3. $||f + g||_p \le ||f||_p + ||g||_p$

     $$\begin{aligned} ||f+g||_p^p &= \int_0^1 |f(x)+g(x)|^p dx = \int_0^1 |f(x)+g(x)||f(x)+g(x)|^{(p-1)} dx \\ &\le \int_0^1 (|f(x)| + |g(x)|)|f(x)+g(x)|^{(p-1)} dx \\ &= \int_0^1 |f(x)||f(x)+g(x)|^{(p-1)} dx + \int_0^1 |g(x)||f(x)+g(x)|^{(p-1)} dx \\ &\le ((\int_0^1 |f(x)|^p dx)^{(1/p)} + (\int_0^1 |g(x)|^p dx)^{(1/p)})(\int_0^1 |f(x)+g(x)|^{(p-1)(p/p-1)} dx)^{(1-1/p)} \end{aligned}$$

(Hölder's inequality)

$= (||f||_p + ||g||_p)\frac{||f+g||_p^p}{||f+g||_p}$

Therefore, $||f+g||_p \leq ||f||_p + ||g||_p$.

Part (b)

If $\{f_k\}$ is Cauchy in L^∞, then for every $m \in \mathbb{N}$ there exists an integer $n \in \mathbb{N}$ such that we have

$$|f_j(x) - f_k(x)| \geq 1/m \text{ for all } j,k \geq n \text{ and } x \in N^c_{j,k,m} \tag{1}$$

where $N_{j,k,m}$ is a null set.

Let $N = \bigcup\limits_{j,k,m \in \mathbb{N}} N_{j,k,m}$. Then N is a null set, and for every $x \in N^c$ the sequence $\{f_k(x) : k \in \mathbb{N}\}$ is Cauchy in $\mathbb{R}$. We define a measurable function $\{f : X \to \mathbb{R}\}$, unique up to a pointwise a.e. equivalence, by

$$f(x) = \lim_{k\to\infty} f_k(x) \text{ for } x \in N^c$$

Letting $k \to \infty$ in (1), we find that for every $m \in \mathbb{N}$ there exists an integer $n \in \mathbb{N}$ such that

$$|f_j(x) - f(x)| \leq 1/m \text{ for } j \geq n \text{ and } x \in N^c \tag{2}$$

It follows that f is essentially bounded and $f_j \to f$ in L^∞ as $j \to \infty$.

This proves that L^∞ is complete.

Part (c)

Andre (solutions to PS#1, problem 2, part b) has given an example for this part. Here I give an example that illustrates the problem with the integral norm more clearly. Set $a = -1$, $b = 1$. Consider the sequence

$$x_n(t) = \begin{cases} -1 & x \in [-1, -\frac{1}{n}) \\ nt & x \in [-\frac{1}{n}, \frac{1}{n}] \\ 1 & x \in (\frac{1}{n}, 1] \end{cases}$$

Note that each function $x_n(t)$ is continuous. We need to verify that the sequence $\{x_n\}$ is Cauchy. WLOG let $n \leq m$.

$$||x_n - x_m|| = \int_{-1}^{1} |x_n(t) - x_m(t)|dt = \int_{-\frac{1}{n}}^{-\frac{1}{m}} |nt+1|dt + \int_{-\frac{1}{m}}^{\frac{1}{m}} |(n-m)t|dt + \int_{\frac{1}{m}}^{\frac{1}{n}} |nt-1|dt = \frac{1}{n} - \frac{1}{m}; \forall n,m \in \mathbb{N}$$

$\forall \epsilon > 0$, $\exists N(\epsilon)\, s.t. \forall n, m > N(\epsilon)$, $||x_n - x_m|| < \epsilon$. (To see this, just take $N(\epsilon) > \frac{1}{\epsilon}$). Therefore, $\{x_n\}$ is Cauchy.

Next, note that this sequence converges pointwise and **w.r.t the integral norm** to

$$x(t) = \begin{cases} -1 & x \in [-1, 0) \\ 0 & x = 0 \\ 1 & x = (0, 1] \end{cases}$$

To see the convergence under the integral norm, consider

$$||x_n - x|| = \int_{-1}^{1} |x_n(t) - x(t)|dt =$$
$$\int_{-1}^{-\frac{1}{n}} |-1-(-1)|dt + \int_{-\frac{1}{n}}^{0} |nt-(-1)|dt + \int_{0}^{\frac{1}{n}} |nt-1|dt + \int_{\frac{1}{n}}^{1} |1-1|dt = \frac{1}{n} \to 0.$$

But the function $x(t)$ is not continuous. Hence although the Cauchy sequence converges, its limit does not belong to the space S.

This example does not work under the sup norm $||x(t)|| = |\sup x(t)|$ because the sequence is not Cauchy. Assume WLOG that $n > m$. We have

$$|x_n(t) - x_m(t)| = \begin{cases} 0 & t \in [-1, -\frac{1}{n}) \\ |nt + 1| & t \in [-\frac{1}{n}, -\frac{1}{m}) \\ |(n-m)t| & t \in [-\frac{1}{n}, -\frac{1}{m}) \\ |nt - 1| & [-\frac{1}{n}, -\frac{1}{m}) \\ 0 & [-\frac{1}{n}, -\frac{1}{m}] \end{cases}$$

Therefore, $||x_n - x_m|| = |\sup x_n(t) - x_m(t)| = \frac{n-m}{m}$. Now take $m = 2n$, we get $||x_n - x_m|| = \frac{1}{2}$. Which shows that the sequence is not Cauchy with respect to the sup norm.

Problem 3

Note that $\phi(x,t) \geq 0$, $\forall x, t$.

Consider the functional form

$$f(x) = \beta \left((u(x))^{\frac{1}{\alpha}} + \left(\int_a^b \phi(x,t) f(t) dt \right)^{\frac{1}{\alpha}} \right)^{\alpha}$$

Note that we can write this equation as

$$(f(x))^{\frac{1}{\alpha}} = \beta^{\frac{1}{\alpha}} \left((u(x))^{\frac{1}{\alpha}} + \left(\int_a^b \phi(x,t) f(t) dt \right)^{\frac{1}{\alpha}} \right)$$

Denote $v(x) := (f(x))^{\frac{1}{\alpha}}$, and define the operator $(Tv)(x) := \beta^{\frac{1}{\alpha}} \left((u(x))^{\frac{1}{\alpha}} + \left(\int_a^b \phi(x,t) v(t)^\alpha dt \right)^{\frac{1}{\alpha}} \right)$ on the space of bounded functions $v : [a,b] \to \mathbf{R}_+$, with the sup norm, $B[0,1]$. To use the contraction mapping theorem, we need two things. First, we need to show that the mapping T is a self-mapping: that is, $T : B[a,b] \to B[a,b]$. By assumption, we know that $u(x) : [a,b] \to \mathbf{R}_+$ is bounded, also $\int_a^b \phi(x,t) v(t)^\alpha dt$ is bounded, given that v is bounded and $\int_a^b \phi(x,t) dt = 1$. Hence, T is a self-mapping.

Secondly, we need to show that T is a contraction. To show this we can use *Blackwell's sufficient conditions for contraction mapping.* The desired properties are the followings:

a. (monotonicity) $v, g \in B[a,b]$ and $v(x) \leq g(x)$, for all $x \in [a,b]$, implies $(Tv)(x) \leq (Tg)(x)$, for all $x \in [a,b]$.

b. (discounting) there exists some $\beta' \in (0,1)$ such that

$$[T(v+c)](x) \leq (Tv))(x) + \beta' c, \qquad \text{all } v \in B[a,b], c \geq 0, x \in [a,b].$$

I begin by showing monotonicity. Indeed if $v(x) \leq g(x)$, $\forall x$, we have $\int_a^b \phi(x,t) v(t)^\alpha dt \leq \int_a^b \phi(x,t) g(t)^\alpha dt$, given that $\phi(x,t) \geq 0$. Next, to show that t satisfies the discounting property, I first show that $||v||_\alpha := \left(\int_a^b \phi(x,t) v(t)^\alpha dt \right)^{\frac{1}{\alpha}}$ defines a norm on the set $B[a,b]$ and hence satisfies the triangle inequality. We need to check three properties:

- $||v||_\alpha \geq 0$ with equality iff $v = 0$
- $||\lambda v||_\alpha = |\lambda| ||v||_\alpha$
- $||v + g||_\alpha \leq ||v||_\alpha + ||g||_\alpha$ (the triangle inequality).

The first two properties are obvious. I only show the triangle inequality:

$$||v+g||_\alpha^\alpha = \int_a^b \phi(x,t)(v(t)+g(t))(v(t)+g(t))^{\alpha-1}dt =$$
$$\int_a^b \phi(x,t)v(t)(v(t)+g(t))^{\alpha-1}dt + \int_a^b \phi(x,t)g(t)(v(t)+g(t))^{\alpha-1}dt =$$
$$\int_a^b \phi(x,t)^{\frac{1}{\alpha}}v(t)\phi(x,t)^{\frac{\alpha-1}{\alpha}}(v(t)+g(t))^{\alpha-1}dt + \int_a^b \phi(x,t)^{\frac{1}{\alpha}}g(t)\phi(x,t)^{\frac{\alpha-1}{\alpha}}(v(t)+g(t))^{\alpha-1}dt \leq$$
$$\left(\left(\int_a^b \left[\phi(x,t)^{\frac{1}{\alpha}}v(t)\right]^\alpha dt\right)^{\frac{1}{\alpha}} + \left(\int_a^b \left[\phi(x,t)^{\frac{1}{\alpha}}g(t)\right]^\alpha dt\right)^{\frac{1}{\alpha}}\right) \times$$
$$\left(\int_a^b \left[\phi(x,t)^{\frac{\alpha-1}{\alpha}}(v(t)+g(t))^{\alpha-1}dt\right]^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}} \qquad \text{(Holder's inequality)}$$
$$= (||v||_\alpha + ||g||_\alpha) \times ||v+g||_\alpha^{\alpha-1}$$

It follows that $||v+g||_\alpha \leq ||v||_\alpha + ||g||_\alpha$.

Next, applying the tringle inequality, I can write

$$\left(\int_a^b \phi(x,t)(v(t)+c)^\alpha dt\right)^{\frac{1}{\alpha}} \leq \left(\int_a^b \phi(x,t)(v(t))^\alpha dt\right)^{\frac{1}{\alpha}} + \left(\int_a^b \phi(x,t)c^\alpha dt\right)^{\frac{1}{\alpha}}$$
$$= \left(\int_a^b \phi(x,t)(v(t))^\alpha dt\right)^{\frac{1}{\alpha}} + c$$

Therefore I have

$$\beta^{\frac{1}{\alpha}}\left((u(x))^{\frac{1}{\alpha}} + \left(\int_a^b \phi(x,t)(v(t)+c)^\alpha dt\right)^{\frac{1}{\alpha}}\right) \leq \beta^{\frac{1}{\alpha}}\left((u(x))^{\frac{1}{\alpha}} + \left(\int_a^b \phi(x,t)(v(t))^\alpha dt\right)^{\frac{1}{\alpha}} + c\right)$$
$$= \beta^{\frac{1}{\alpha}}\left((u(x))^{\frac{1}{\alpha}} + \left(\int_a^b \phi(x,t)(v(t))^\alpha dt\right)^{\frac{1}{\alpha}}\right) + \beta^{\frac{1}{\alpha}}c = \beta^{\frac{1}{\alpha}}\left((u(x))^{\frac{1}{\alpha}} + \left(\int_a^b \phi(x,t)(v(t))^\alpha dt\right)^{\frac{1}{\alpha}}\right) + \beta' c$$

showing that T is a contraction with modulus β'. Lastly, note that the operator T maps the subset $C[a,b] \subseteq B[a,b]$ of continuous bounded functions to itself and therefore we can apply the contraction mapping theorem (remember that Contraction mapping theorem requires a complete metric space). Thus the system has a unique fixed point $v^* \in C[a,b]$.

Problem 4

Part (a)

Note first of all that in a sequential equilibrium, the budget constraint is homogeneous of degree 0 in prices and we can normalize $p_{c,t} = 1$ (or if you prefer, we can divide by $p_{c,t}$ and write our other prices as ratios). Hence we can write the household's problem (suppressing notation for j) as

$$\max_{\{c_t,x_t,k_{t+1},a_{t+1},l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t,l_t) \ s.t.$$

$$\begin{aligned} & c_t + p_{x,t}x_t + q_t a_t \leq w_t(e_t - l_t) + r_t k_t + a_t \ \forall \ t \\ & x_t = k_{t+1} - (1-\delta)k_t \\ & a_t \geq -A \ \forall \ t \\ & a_0, k_0 \text{ given} \end{aligned} \tag{3}$$

For feasibility we still need the goods, labor, and capital markets to clear, but now we have the additional constraint that

$$\sum_j a_t^j = 0$$

Intuitively, for every borrower we need a lender, so that in the aggregate household debt sums to zero. Recall Walras' law: if every market but one clears, then all markets clear. Hence if the goods markets clear, then the lending market has to clear, which means that borrowing = lending.

Part (b)

A competitive equilibrium is allocations for the household $\{c_t^j, x_t^j, l_t^j, k_t^j, a_t^j\}_{j=1,\ldots,J}^{t=0,\ldots,\infty}$ and firm $\{y_t^f, k_t^f, n_t^f\}_{f=c,x}^{t=0,\ldots,\infty}$ and prices $\{q_t, p_{x,t}, r_t, w_t\}_{t=0}^{\infty}$ such that

- Households solve (3)
- The representative firms (simplifying here) solve

$$\max_{y_t,k_t,l_t} p_{t,f}y_t^f - w_t n_t^f - r_t k_t^f \ s.t.$$
$$y_t^f = F^f(k_t^f, n_t^f)$$

for $f \in \{c, x\}$

- Markets clear:

$$\sum_j c_t^j = y_t^c$$
$$\sum_j x_t^j = y_t^x$$
$$\sum_j (e_t^j - l_t^j) = \sum_f n_t^f$$
$$\sum_j k_t^j = \sum_f k_t^f$$
$$\sum_j a_t^j = 0 \qquad \forall\ i$$

- Profits are consistent with firm behavior (here excluded since profits are zero at the CE)

Part (c)

We want to show that the Arrow-Debreu budget constraint is equivalent to the budget constraint with sequential markets. The AD budget constraint is

$$\sum_{t=0}^{\infty} (p_{c,t}c_t + p_{x,t}x_t) \leq \sum_{t=0}^{\infty} (w_t(e_t - l_t) + r_t k_t)$$
$$\implies \sum_{t=0}^{\infty} (p_{c,t}c_t + p_{x,t}k_{t+1}) \leq \sum_{t=0}^{\infty} (w_t(e_t - l_t) + r_t k_t + p_{x,t}(1-\delta)k_t)$$

Consider the FOC for this problem:

$$\beta^t u_c(c_t, l_t) = p_{c,t}\lambda$$
$$\beta^t u_l(c_t, l_t) = w_t\lambda$$
$$p_{x,t} = (r_{t+1} + p_{x,t+1}[1-\delta])$$

Note that from the firm's problem, we have $r_t = p_{x,t}F_k(k_t^x, n_t^x)$. Therefore we can rewrite and combine the FOC to show that, at an interior CE,

$$u_c(c_t, l_t) = \frac{p_{c,t}}{p_{c,t+1}}\beta u_c(c_{t+1}, l_{t+1})$$
$$u_c(c_t, l_t) = \frac{p_{c,t}}{w_t}u_l(c_t, l_t)$$
$$\frac{p_{x,t}}{p_{x,t+1}} = (F_k(k_{t+1}^x, n_{t+1}^x) + 1 - \delta)$$
$$= \frac{p_{c,t}}{p_{c,t+1}}$$

where the last equality follows from the fact that both firms pay the same rental rate for capital (otherwise one firm would be unable to rent capital!). Now turning our attention to the sequential markets economy and taking the FOC,

$$\begin{aligned}\beta^t u_c(c_t, l_t) &= \lambda_t \\ \beta^t u_l(c_t, l_t) &= w_t \lambda_t \\ \lambda_t p_{x,t} &= \lambda_{t+1} \left(r_{t+1} + p_{x,t+1}(1-\delta)\right) \\ \lambda_t q_t &= \lambda_{t+1}\end{aligned}$$

We can rewrite these conditions as

$$\begin{aligned}u_c c_t, l_t &= \frac{\lambda_t}{\lambda_{t+1}} \beta u_c(c_{t+1}, n_{t+1}) \\ u_c(c_t, l_t) &= \frac{1}{w_t} u_l(c_t, l_t) \\ \frac{\lambda_t}{\lambda_{t+1}} &= \frac{p_{x,t+1}}{p_{x,t}}(F_k(k^x_{t+1}, n^x_{t+1}) + 1 - \delta) \\ &= (F_k(k^x_{t+1}, n^x_{t+1}) + 1 - \delta) \qquad \text{(why?)} \\ &= \frac{1}{q_t}\end{aligned}$$

But then we have shown that

$$\begin{aligned}\text{AD} : u_c(c_t, l_t) &= \beta u_c(c_{t+1}, l_{t+1})(F_k(k^x_{t+1}, n^x_{t+1}) + 1 - \delta) \\ \text{Sequential} : u_c(c_t, l_t) &= \beta u_c(c_{t+1}, l_{t+1})(F_k(k^x_{t+1}, n^x_{t+1}) + 1 - \delta)\end{aligned}$$

If we perform the same exercise with the *intratemporal* Euler equation (i.e. the labor-leisure trade-off), we will obtain the same result: the Euler equations for these two economies are identical. Moreover, we know that because the firms' problem is the same in both economies, the firms' allocations will be the same; and we have therefore shown that the CE in the Arrow-Debreu economy is equivalent to the CE in the sequential markets economy, and the relationship between prices in these two economies is $\frac{1}{q_t} = r_{t+1} + 1 - \delta$.

Part (d)

From above, and supposing we hadn't normalized prices, we would have

$$\begin{aligned} p_{x,t} &= \frac{\lambda_{t+1}}{\lambda_t}\left(r_{t+1} + p_{x,t+1}(1-\delta)\right) \\ &= \frac{q_t}{p_{c,t+1}}\left(r_{t+1} + p_{x,t+1}(1-\delta)\right) \\ &= p_{c,t}\left(r_{t+1} + p_{x,t+1}(1-\delta)\right) \\ &= r_{t+1} + p_{x,t+1}(1-\delta) \end{aligned}$$

or rearranging,

$$r_{t+1} - p_{x,t} + p_{x,t+1}(1-\delta) = 0$$

In words, the rental value of capital is offset by the capital gains loss. If this equality didn't hold - for instance, if $r_{t+1} > p_{x,t} - p_{x,t+1}(1-\delta)$ - then the household could purchase a unit of capital today for $p_{x,t}$ and tomorrow receive $p_{x,t+1}(1-\delta) + r_{t+1}$, yielding positive profits after repaying $p_{x,t}$. To see how r_t relates to q_t, also from above (and with normalized prices) we have

$$\frac{1}{q_{t-1}} = r_{t+1} + (1-\delta)$$

or the return on bonds (LHS) is equal to the return on capital (RHS). Here as well, if the equality didn't hold then we would have an arbitrage opportunity: sell short bonds and invest, or sell short next-period capital and buy bonds, depending upon which return is lower.

Problem 5

Part (a)

A competitive equilibrium is allocations for the households $\{c_i\}_{i=1}^N$ and firms $\{K_i, L_i, Q_{i1}, ..., Q_{iN}\}_{i=1}^N$ and prices $\{r, w, p_i\}_{i=1}^N$ such that

- Households solve

$$\max \sum_{i=1}^N \alpha_i \log c_i \ s.t.$$
$$\sum_{i=1}^N p_i c_i \leq rk + wl$$

  where $k = \bar{K}/I$ and $l = \bar{L}/I$.

- Firms solve

$$\max\ p_iY_i - rK_i - wL_i - \sum_{j=1}^{N} p_jQ_{ij}\ s.t.$$

$$Y_i = A_iK_i^{\alpha}L_i^{\gamma-\alpha}\prod_{j=1}^{N} Q_{ij}^{(1-\gamma)\omega_{ij}}$$

- Markets clear:

$$\bar{C}_i + \sum_{j=1}^{N} Q_{ji} = Y_i\ \forall\ i$$

$$\bar{K} = \sum_{j=1}^{N} K_j$$

$$\bar{L} = \sum_{j=1}^{N} L_j$$

Part(b)

Taking the FOC of the firm's cost-minimization problem, we get

$$r = \lambda\alpha A_iK_i^{\alpha-1}L_i^{\gamma-\alpha}\prod_{j=1}^{N} Q_{ij}^{(1-\gamma)\omega_{ij}}$$

$$w = \lambda(\gamma-\alpha)A_iK_i^{\alpha}L_i^{\gamma-\alpha-1}\prod_{j=1}^{N} Q_{ij}^{(1-\gamma)\omega_{ij}}$$

$$p_j = \lambda(1-\gamma)\omega_{ij}A_iK_i^{\alpha}L_i^{\gamma-\alpha}Q_{ij}^{(1-\gamma)\omega_{ij}-1}\prod_{k\neq j} Q_{ik}^{(1-\gamma)\omega_{ik}}$$

The simplest way to solve this is by multiplying each side by the factor of production for which we took the derivative, and re-obtaining Y:

$$K_i\frac{r}{\alpha} = Y_i\lambda$$

$$L_i\frac{w}{\gamma-\alpha} = Y_i\lambda$$

$$Q_{ij}\frac{p_j}{(1-\gamma)\omega_{ij}} = Y_i\lambda$$

It follows that

$$\begin{aligned}
L_i &= \frac{r(\gamma-\alpha)}{\alpha w} K_i \\
Q_{i,j} &= \frac{r(1-\gamma)\omega_{ij}}{\alpha p_j} K_i \\
\implies Y_i &= A_i \left(\frac{r(\gamma-\alpha)}{\alpha w}\right)^{\gamma-\alpha} \prod_{j=1}^{N} \left(\frac{r(1-\gamma)\omega_{ij}}{\alpha p_j}\right)^{(1-\gamma)\omega_{ij}} K_i \\
\implies K_i &= A_i^{-1} \left(\frac{r(\gamma-\alpha)}{\alpha w}\right)^{\alpha-\gamma} \prod_{j=1}^{N} \left(\frac{r(1-\gamma)\omega_{ij}}{\alpha p_j}\right)^{(\gamma-1)\omega_{ij}} Y_i \\
&= A_i^{-1} \left(\frac{r}{\alpha}\right)^{\alpha-1} \left(\frac{\gamma-\alpha}{w}\right)^{\alpha-\gamma} \prod_{j=1}^{N} \left(\frac{p_j}{(1-\gamma)\omega_{ij}}\right)^{(1-\gamma)\omega_{ij}} Y_i \\
&= P_i^k Y_i
\end{aligned}$$

Using the same steps we can show that each input is some constant fraction of Y:

$$\begin{aligned}
L_i &= A_i^{-1} \left(\frac{\alpha}{r}\right)^{-\alpha} \left(\frac{w}{\gamma-\alpha}\right)^{\gamma-\alpha-1} \prod_{j=1}^{N} \left(\frac{p_j}{(1-\gamma)\omega_{ij}}\right)^{(1-\gamma)\omega_{ij}} Y_i = P_i^l Y_i \\
Q_{ij} &= A_i^{-1} \left(\frac{\alpha}{r}\right)^{-\alpha} \left(\frac{\gamma-\alpha}{w}\right)^{\alpha-\gamma} \prod_{k\neq j} \left(\frac{p_k}{(1-\gamma)\omega_{ik}}\right)^{(1-\gamma)\omega_{ij}} \left(\frac{p_j}{(1-\gamma)\omega_{ij}}\right)^{(\gamma-1)\omega_{ii}-1} Y_i = P_i^j Y_i
\end{aligned}$$

But then, returning to the cost function,

$$\begin{aligned}
C_i(Y) &= r P_i^k Y_i + w P_i^l Y_i + \sum_{j=1}^{N} p_j P_i^j Y_i \\
&= \psi Y_i
\end{aligned}$$

as we wanted to show.

Part (c)

We know that at any solution to the firm's problem, costs are a constant proportion of output - i.e. the proportion of inputs is static and the production function is homogeneous of degree 0. We can therefore consider a 'reduced form' profit function for the firm:

$$\begin{aligned}
\pi(Y_i; p) &= p_i Y_i - C(Y_i; p) \\
&= p_i Y_i - \psi_i Y_i
\end{aligned}$$

where p is the vector of prices. The first-order condition is

$$p_i - \psi_i = 0$$

which states that the firm should produce up to the point that marginal revenue (p_i) equals marginal cost (ψ_i).

Part (d)

We now have the system of equations

$$\begin{aligned} p_i &= rP_i^k + wP_i^l + \sum_{j=1}^{N} p_j P_i^j \\ &= A_i^{-1}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{\gamma-\alpha}\right)^{\gamma-\alpha}\prod_{j=1}^{N}\left(\frac{p_j}{(1-\gamma)\omega_{ij}}\right)^{(1-\gamma)\omega_{ij}}\left[\alpha+(\gamma-\alpha)+\sum_{j=1}^{N}\omega_{ij}(1-\gamma)\right] \\ &= A_i^{-1}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{\gamma-\alpha}\right)^{\gamma-\alpha}\left(\frac{1}{1-\gamma}\right)^{1-\gamma}\prod_{j=1}^{N}\left(\frac{p_j}{\omega_{ij}}\right)^{(1-\gamma)\omega_{ij}} \end{aligned}$$

Taking logs,

$$\ln(p_i) = -\ln A_i + \ln\left(\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{\gamma-\alpha}\right)^{\gamma-\alpha}\right) + \sum_{j=1}^{N}(1-\gamma)w_{ij}\ln p_j - \sum_{j=1}^{N}(1-\gamma)w_{ij}\ln((1-\gamma)w_{ij})$$

Switching to matrix notation and letting $\hat{p} = [\log p_1, ..., \log p_N]$, $\hat{A} = [\ln A_1, ..., \ln A_N]$, and Γ correspond to the matrix of input-output elasticities,

$$\begin{aligned} \hat{p} &= -\hat{A} + \vec{Q} + (1-\gamma)\Gamma\hat{p} \\ &= (I-(1-\gamma)\Gamma)^{-1}(-\hat{A}+\vec{Q}) \end{aligned}$$

where $Q_i = \alpha\ln\left(\frac{r}{\alpha}\right) + (\gamma-\alpha)\ln\left(\frac{w}{\gamma-\alpha}\right) - \sum_{j=1}^{N}(1-\gamma)\omega_{ij}\log([1-\gamma]\omega_{ij})$.

Part (e)

We normalize $w = 1$, in which case from the firm's FOC we have

$$r = \frac{\alpha}{\gamma-\alpha}\frac{L_i}{K_i}$$

The market-clearing conditions for capital and labor are

$$\sum_i K_i = \bar{K}$$

$$\sum_i L_i = \bar{L}$$

and since each firm uses capital and labor in the same proportion, we have

$$r = \frac{\alpha}{\gamma - \alpha} \frac{\bar{L}}{\bar{K}} \tag{4}$$

The price equation $\hat{p}$ now simplifies to

$$\hat{p} = (I - (1-\gamma)\Gamma)^{-1} \left(-\hat{A} + \left(\gamma \ln w + \alpha \ln \left(\frac{\bar{L}}{\bar{K}} \right) \right) \vec{e} - \vec{Q}' \right)$$

where $\vec{e} = \begin{bmatrix} 1 \\ ... \\ 1 \end{bmatrix}$ and $\vec{Q}' = -\gamma \ln(\gamma - \alpha) - \sum_{j=1}^{N} (1-\gamma) w_{ij} \ln((1-\gamma) w_{ij})$, and hence we have solved for all prices as functions of the elasticity terms, productivity, and the aggregate endowments of capital and labor.

To solve for production, we need to turn to the household's problem. The household's FOC are

$$\frac{\alpha_i}{p_i c_i} = \frac{\alpha_j}{p_j c_j}$$
$$\implies c_i = \frac{\alpha_i p_j}{\alpha_j p_i} c_j$$

The household's budget constraint is

$$\sum_i p_i c_i \leq rk + wl$$

which will hold with equality at the optimum. Assuming for simplicity that $\sum \alpha_i = 1$ and solving for demand,

$$\begin{aligned} c_i &= \frac{rk + wl}{p_i} - \sum_{j \neq i} \frac{p_j c_j}{p_i} \\ &= \frac{rk + wl}{p_i} - \sum_{j \neq i} \frac{\alpha_j}{\alpha_i} c_i \\ &= \frac{rk + wl}{p_i} - \frac{1 - \alpha_i}{\alpha_i} c_i \\ &= \frac{\alpha_i}{p_i} (rk + wl) \\ &= \frac{\alpha_i}{p_i} \left(\frac{\gamma}{\gamma - \alpha} \bar{L} \right) \end{aligned}$$

For each good, we have the market clearing equation

$$\sum_i Q_{ij} + c_j = Y_j$$

Substituting back into the market-clearing condition,

$$\begin{aligned}
\sum_i Q_{ij} + \frac{\alpha_i}{p_i}(r\bar{K} + w\bar{L}) &= Y_j \\
\implies \sum_i P_i^j Y_i + \frac{\alpha_i}{p_i}\left(\frac{\gamma}{\gamma-\alpha}\bar{L}\right) &= Y_j \\
\implies Y = (I - P')^{-1} d \left(\frac{\gamma}{\gamma-\alpha}\bar{L}\right)
\end{aligned}$$

where $d_i = \frac{\alpha_i}{p_i}$ and P is the matrix of P_i^j defined earlier.

Part (f)

Nominal GDP is (not assuming $w = 1$)

$$\begin{aligned}
GDP &= \sum_i p_i c_i = \sum_i p_i d_i (r\bar{K} + w\bar{L}) \\
&= \sum_i \left(\frac{p_i d_i \gamma w}{\gamma - \alpha}\bar{L}\right) \\
&= \frac{\gamma w}{\gamma - \alpha}\bar{L}
\end{aligned}$$

However, in this question we want to solve for the real GDP. Therefore, we need first to find the price index. With the Cobb-Douglas utility function the price index is given by $\prod_{i=1}^N p_i^{\alpha_i}$. We normalize this index to 1, meaning $\log\left(\prod_{i=1}^N p_i^{\alpha_i}\right) = \sum_{i=1}^N \alpha_i \log p_i = \alpha^T \hat{p} = 0$. Now replacing from $\hat{p}$ (part d) we get

$$\alpha^T \hat{p} = \alpha^T (I - (1-\gamma)\Gamma)^{-1}\left(-\hat{A} + \left(\gamma \ln w + \alpha \ln\left(\frac{\bar{L}}{\bar{K}}\right)\right)\vec{e} - \vec{Q}'\right) = 0$$

Next, note that $\Gamma\vec{e} = \vec{e}$, (because $\sum_{j=1}^N w_{ij} = 1, \forall i$), therefore $\vec{e} - (1-\gamma)\Gamma\vec{e} = \gamma\vec{e}$, and we have $(I - (1-\gamma)\Gamma)\vec{e} = \gamma\vec{e}$. Hence, $(I - (1-\gamma)\Gamma)^{-1}\vec{e} = \gamma^{-1}\vec{e}$. We get

$$\begin{aligned}
\alpha^T \hat{p} &= \alpha^T (I - (1-\gamma)\Gamma)^{-1}\left(-\hat{A} - \vec{Q}'\right) + \ln w + \alpha \ln\left(\frac{\bar{L}}{\bar{K}}\right) = 0 \rightarrow \\
\ln w &= \alpha^T (I - (1-\gamma)\Gamma)^{-1}\left(\hat{A} + \vec{Q}'\right) - \alpha \ln\left(\frac{\bar{L}}{\bar{K}}\right) \rightarrow \\
w &= C\left(\frac{\bar{L}}{\bar{K}}\right)^{-\alpha}
\end{aligned}$$

Replacing this in nominal GDP gives us the real GDP:

$$GDP_r = \frac{\gamma}{\gamma - \alpha} C \left(\frac{\bar{L}}{\bar{K}}\right)^{-\alpha} \bar{L} = \frac{\gamma}{\gamma - \alpha} C \bar{L}^{1-\alpha}\bar{K}^{\alpha}$$

part (g)

Following the notation in previous parts

$$Y_i = \frac{\alpha_i}{p_i}\frac{\gamma w\bar{L}}{\gamma-\alpha} + (1-\gamma)\sum_{j=1}^{N} w_{ji}\frac{p_jY_j}{p_i}$$

$$\rightarrow p_iY_i - (1-\gamma)\sum_{j=1}^{N} w_{ji}p_jY_j = \alpha_i\frac{\gamma w}{\gamma-\alpha}\bar{L}$$

In matrix form

$$X = [I-(i-\gamma)\Gamma^T]^{-1}\vec{\alpha}\frac{\gamma w}{\gamma-\alpha}\bar{L}$$

where $X = \begin{bmatrix} p_1Y_1 \\ ... \\ p_NY_N \end{bmatrix}$.

To get the real aggregate output we multiply the vector X by $\vec{e}^T$ from left and substitute the normalized wage derived in part f

$$\vec{e}^T X = \gamma^{-1}\vec{e}^T\vec{\alpha}\frac{\gamma w}{\gamma-\alpha}\bar{L} = \frac{w}{\gamma-\alpha}\bar{L} = \frac{C}{\gamma-\alpha}\bar{L}^{1-\alpha}\bar{K}^{\alpha}$$

Now consider a change of $g\%$ in A_i

$$C' = \exp\left[\alpha^T(I-(1-\gamma)\Gamma)^{-1}(\hat{A}+\ln(1+g)\vec{e_i}+\vec{Q'})\right] \quad = C\times\exp[\alpha^T(I-(1-\gamma)\Gamma)^{-1}\ln(1+g)\vec{e_i}]$$

Note that if we assume the productivity of all sectors in the economy is changing, instead of $\vec{e_i}$ we will have the vector $\vec{e}$. In that case $[\alpha^T(I-(1-\gamma)\Gamma)^{-1}\ln(1+g)\vec{e}]$ simplifies to $\gamma^{-1}\ln(1+g)$ (note that I have shown in part f that $\vec{e}$ is an eigenvector of Γ). Hence

$$C' = C\times(1+g)^{\frac{1}{\gamma}} \rightarrow$$

$$\vec{e}^T X = \frac{C(1+g)^{\frac{1}{\gamma}}}{\gamma-\alpha}\bar{L}^{1-\alpha}\bar{K}^{\alpha}$$

Therefore, we see that a $g\%$ change in all A_i's leads to a change of $(\frac{1}{\gamma}g\%)$ in aggregate output. γ here determines the magnitude of the effect. In fact γ captures the extend to which the economy depends on the network structure for production versus mere dependence on capital and labor. Now suppose only 1 of the sectors (sector i) changes. We get

$$[\alpha^T(I-(1-\gamma)\Gamma)^{-1}\vec{e_i}\ln(1+g)] = [\alpha^T(\sum_{k=0}^{\infty}(1-\gamma)^k\Gamma^k)\vec{e_i}\ln(1+g)]$$

$$= \ln(1+g)\left(1+\sum_{j=1}^{N}\alpha_j w_{ji} + \alpha^T(\sum_{k=0}^{\infty}(1-\gamma)^k\Gamma^k)\vec{e_i}\right)$$

I expand the series just to show what kind of terms show up in The $\alpha^T(I-(1-\gamma)\Gamma)^{-1}\vec{e_i}$. For example the second term ($\sum_{j=1}^N \alpha_j w_{ji}$) illustrates that the effect of a shock to sector i depends (among other things) on the number of nodes to which it is directly connected and also the magnitude of this connection denoted by w_{ji}. Generally, a change of $g\%$ in A_i leads to a change in aggregate output wich depends on the terms in the i'th column of $\alpha^T(I-(1-\gamma)\Gamma)^{-1}$ (which captures a measure of centrality for sector i).

Part (h)

Compare the two matrices

$$W_c = \begin{bmatrix} 1 & 0 & ... & ... & 0 & 0 \\ 1 & 0 & ... & ... & 0 & 0 \\ 0 & 1 & ... & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & 0 & 0 \\ 0 & 0 & ... & ... & 1 & 0 \end{bmatrix}$$

and

$$W_s = \begin{bmatrix} 1 & 0 & ... & ... & 0 & 0 \\ 1 & 0 & ... & ... & 0 & 0 \\ 1 & 0 & ... & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 1 & 0 & ... & ... & 0 & 0 \\ 1 & 0 & ... & ... & 0 & 0 \end{bmatrix}$$

In the first case, aggregate production function looks like

$$p.Y = p_1 A_1 K_1^{\alpha} L_1^{\gamma-\alpha} Q_{11}^{1-\gamma} + \sum_{i\geq 2} p_i A_i K_i^{\alpha} L_i^{\gamma-\alpha} Q_{i,i-1}^{1-\gamma}$$

and in the second case,

$$p.Y = \sum_{i\geq 1} p_i A_i K_i^{\alpha} L_i^{\gamma-\alpha} Q_{i1}^{1-\gamma}$$

In both cases, sector $i=1$ is critical; any change in the behavior of this sector will have an 'outsized' effect on aggregate production. In the case of a chain network, other sectors may also be critical: if something (e.g. a productivity shock) affects sector k, then every sector with $i>k$ will also be affected. In the case of a star network, however, if a sector other than sector 1 is affected, other sectors are only affected indirectly through the reduction in the demand for the product of sector 1 (which may affect the price of good 1).

Problem 6

In this problem, we develop a model where health status depends on age and time. Let $h^s_{a,t}$ and $h^e_{a,t}$ denote the health status and health expenditure of an individual of age a at time t. Let $N_{a,t}$ denote the number of people of age a at time t. Mortality rate can be defined as the inverse of an individual's health status and hence, the probability of survival can be calculated as $1 - 1/h^s_{a,t}$. A competitive equilibrium is a sequence of allocations $\{c_{a,t}, l_{a,t}, k_{a,t+1}, h^e_{a,t}\}$, output $\{y^c_t, k^c_{a,t}, n^c_{a,t}\}$, $\{y^k_t, k^k_{a,t}, n^k_{a,t}\}$, $\{y^h_t, k^h_{a,t}, n^h_{a,t}\}$, and prices $\{p_{c,t}, p_{x,t}, p_{h,t}, r_t, w_t\}$ such that:

1. Households maximize utility taking prices as given:

$$\begin{aligned}
\max_{\{c_{a,t}, l_{a,t}, k_{a,t+1}, h^e_{a,t}\}} \sum_{t=0}^{\infty}\sum_{a=0}^{\infty} & N_{a,t}\beta^t u(c_{a,t}, h^s_{a,t}, l_{a,t}) \ s.t. \\
& N_{a,t}[p_{c,t}c_{a,t} + p_{x,t}x_t + p_{h,t}h^e_{a,t}] \leq r_t k_{a,t} + w_t(e_{a,t} - l_{a,t}) \\
& h^s_{a,t} = f(h^e_{a,t}; a, t) \\
& N_{a+1,t+1} = (1 - 1/h^s_{a,t})N_{a,t} \\
& x_{a,t} = k_{a,t+1} - (1-\delta)k_{a,t} \\
& k_{a,0}, N_{0,t} \text{ given}
\end{aligned} \tag{5}$$

2. Firms maximize profits taking prices as given:

$$\begin{aligned}
\text{Consumption Good Producers:} \quad & \max_{n^c_{a,t}, k^c_{a,t}} p_{c,t}F^c(k^c_{a,t}, n^c_{a,t}) - w_t n^c_{a,t} - r_t k^c_{a,t} \\
\text{Investment Good Producers:} \quad & \max_{n^k_{a,t}, k^k_{a,t}} p_{k,t}F^k(k^k_{a,t}, n^k_{a,t}) - w_t n^k_{a,t} - r_t k^k_{a,t} \\
\text{Healthcare Good Producers:} \quad & \max_{n^h_{a,t}, k^h_{a,t}} p_{h,t}F^h(k^h_{a,t}, n^h_{a,t}) - w_t n^h_{a,t} - r_t k^h_{a,t}
\end{aligned} \tag{6}$$

3. Markets Clear:

$$\sum_i [N_{a,t}(c_{a,t} + h_{a,t} + x_{a,t})] = Y_t^c + Y_t^k + Y_t^h$$
$$\sum_i k_{a,t} = k_{a,t}^c + k_{a,t}^k + k_{a,t}^h$$
$$\sum_i (e_{a,t} - l_{a,t}) = n_{a,t}^c + n_{a,t}^k + n_{a,t}^h \tag{7}$$

Household preferences are additively separable and will now depend on consumption, leisure and health status. However, the assumption of additive separability suggests that an individual's marginal utility from consumption and leisure will not depend health status which is counterintuitive. In particular, we expect marginal utility from consumption and leisure to be lower for individuals with a lower/worse health status. Households incur health expenditures to improve their health status and to increase their utility from consumption and leisure. We expect health expenditures to increase with income. For reference, see Jones (2004) and Hall and Jones (2007).

Problem 7

In this problem we extend the model to capture the interaction of environmental factors with the economy. A simple channel for this interaction is that firms pollute the environment as a by-product of the production process, and households are adversely affected by (i.e. get disutility from) pollution. To capture this aspect, we modify the utility functioin of agents to $U(\{c_t^i, l_t^i, E_t\}_t)$ with $\frac{\partial U}{\partial E_t} \leq 0$, $\forall t$, where E_t is the aggregate pollution. We assume $E_t = f(Y_t)$, with $\frac{df}{dY_t} \geq 0$, $\forall t$, indicating that higher levels of production bring about more pollution.

A competitive equilibrium is a sequence of allocations $\{c_t^i, l_t^i, k_{t+1}^i, E_t^i\}_{i,t}$, output $\{y_t, k_t, n_t\}$, and prices $\{p_t, r_t, w_t\}$ such that:

1. Households maximize utility taking prices as given:

$$\max_{\{c_t^i, l_t^i, k_{t+1}^i\}} \sum_{t=0}^{\infty} \beta^t u(c_t^i, l_t^i, E_t) \ s.t.$$
$$\sum_{t=0}^{\infty} p_t(c_t^i + x_t^i) \leq \sum_{t=0}^{\infty} r_t k_t^i + w_t(e_t^i - l_t^i)$$
$$x_t^i = k_{t+1}^i - (1-\delta)k_t^i$$
$$k_0^i \text{ given}$$

Note that households do not have the pollution as a choice variable, because each individual is infinitesimal compared to the whole population, and therefore cannot affect the total output-and pollution-by its consumption and leisure choices.

2. Firms maximize profits taking prices as given:

$$\max_{n_t,k_t} p_t Y_t - w_t n_t^f - r_t k_t^f \ s.t.$$
$$Y_t = F(k_t^f, n_t^f)$$

3. Markets Clear:

$$\sum_i c_t + x_t^i = Y_t$$
$$\sum_i k_t^i = k_t^f$$
$$\sum_i (e_t^i - l_t^i) = n_t^f \tag{8}$$

Here the C.E. outcome is not pareto efficient. Because pollution is a negative externality, meaning that firms do not bear the cost of the pollution, but society does, resulting in excess production. To see this, note that a social planner would solve the following problem instead

$$\begin{aligned}
\max_{\{C_t,L_t,K_{t+1},E_t\}} \sum_{t=0}^{\infty} &\beta^t u(c_t, l_t, E_t) \ s.t. \\
&C_t^i + K_{t+1} - (1-\delta)K_t \leq F(K_t, n_t) \\
&\sum_i (e_t^i - l_t^i) = n_t \\
&E_t = f(F(K_t, n_t)) \\
&K_0 \text{ given}
\end{aligned} \tag{9}$$

Which results in a lower level of production.